A remark on perturbations of sine and cosine sums

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Consider a collection $\lambda_1 < \cdots < \lambda_N$ of distinct positive integers and the quantities

$$M_1 = M_1(\lambda_1, \ldots, \lambda_N) = \max_{0 \le x \le 2\pi} \left| \sum_{j=1}^N \sin \lambda_j x \right|$$

and

$$M_2 = M_2(\lambda_1, \ldots, \lambda_N) = - \min_{0 \le x \le 2\pi} \sum_{j=1}^N \cos \lambda_j x.$$

One is interested in frequencies λ_j which make the quantities M_1 and M_2 small as $N \to \infty$. It is trivial that $M_1 \ge cN^{1/2}$ but it is much harder to show even that $M_2 \to \infty$. (In this note c denotes an absolute positive constant, not necessarily the same in all its occurences.)

It is a result of Bourgain [1, 2] that M_1 may become $O(N^{2/3})$ and it is very easy to construct a collection λ_j which gives $M_2 = O(N^{1/2})$, which is the conjectured optimal. For minimizing M_2 it is also possible to have the collection of frequencies relatively well packed, that is with $\lambda_N \le 2N$ [3], while for any $\epsilon > 0$ and for any collection λ_j that makes $M_1 = O(N^{1-\epsilon})$ one can easily see that λ_N is super-polynomial in N.

Prompted by a discussion with G. Benke we prove that collections of frequencies λ_j which have $M_1 = o(N)$ or $M_2 = o(N)$ are unstable, in the sense that one can perturb the λ_j by one each and get $M_1 \ge cN$ and $M_2 \ge cN$.

Theorem 1 (i) *Suppose that $M_1(\lambda_1, \ldots, \lambda_N) = o(N)$. Then there exists a choice of $\epsilon_j = \pm 1$, $j = 1, \ldots, N$, so that $M_1(\lambda_1 + \epsilon_1, \ldots, \lambda_N + \epsilon_N) \ge cN$.*
(ii) *Suppose, similarly, that $M_2(\lambda_1, \ldots, \lambda_N) = o(N)$. Then there exists a choice of $\epsilon_j = \pm 1$, $j = 1, \ldots, N$, so that $M_2(\lambda_1 + \epsilon_1, \ldots, \lambda_N + \epsilon_N) \ge cN$.*

Remark. It is not always the case that the perturbed frequencies are all distinct but is frequently so and, in any case, at most two may overlap at any given integer.

Proof. (i) Write

$$\begin{aligned}
\sum_{j=1}^N \sin{(\lambda_j + \epsilon_j)x} &= \sum_{j=1}^N \sin \lambda_j x \cos \epsilon_j x + \sum_{j=1}^N \cos \lambda_j x \sin \epsilon_j x \\
&= \cos x \sum_{j=1}^N \sin \lambda_j x + \sin x \sum_{j=1}^N \epsilon_j \cos \lambda_j x \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

We have $\mathrm{I} = o(N)$. For $\lambda > 10$, say, we have

$$\frac{4}{\pi} \int_{\pi/4}^{\pi/2} \left| \cos \lambda x \right| \, dx \ge c.$$

From this we deduce that

$$\frac{4}{\pi} \int_{\pi/4}^{\pi/2} \sum_{j=1}^N \left| \cos \lambda_j x \right| \, dx \ge cN,$$

hence there exists $x_0 \in [\frac{\pi}{4}, \frac{\pi}{2}]$ such that

$$\sum_{j=1}^{N} |\cos \lambda_j x_0| \geq cN.$$

Choose then $\epsilon_j = \mathrm{sgn}\,(\cos \lambda_j x_0)$ to get

$$\sum_{j=1}^{N} \epsilon_j \cos \lambda_j x_0 \geq cN.$$

Since $\sin x \geq 2^{-1/2}$ in $[\frac{\pi}{4}, \frac{\pi}{2}]$ we get that $\mathrm{II} \geq cN$ at x_0, which gives the required $M_1(\lambda_1+\epsilon_1, \ldots, \lambda_N+\epsilon_N) \geq cN$, since $\mathrm{I} = o(N)$ everywhere.

(ii) The proof is similar. We write

$$\begin{aligned} \sum_{j=1}^{N} \cos(\lambda_j+\epsilon_j)x &= \cos x \sum_{j=1}^{N} \cos \lambda_j x - \sin x \sum_{j=1}^{N} \epsilon_j \sin \lambda_j x \\ &= \mathrm{I} - \mathrm{II}. \end{aligned}$$

For $x \in [\frac{4\pi}{6}, \frac{5\pi}{6}]$ we have $\mathrm{I} \leq o(N)$ so it is enough to show that for some x_0 in the same interval $\mathrm{II} \geq cN$. To do that observe, as before, that

$$\frac{6}{\pi} \int_{4\pi/6}^{5\pi/6} \sum_{j=1}^{N} |\sin \lambda_j x| \geq cN, \tag{1}$$

and choose $\epsilon_j = \mathrm{sgn}\,(\sin \lambda_j x_0)$, where x_0 is a point that makes the left hand side of (1) large. This implies the existence of $x_0 \in [\frac{4\pi}{6}, \frac{5\pi}{6}]$ such that $\mathrm{II} \geq cN$ at x_0.
$\Box$

Take a Bourgain's collection of frequencies λ_j for which $M_1 = O(N^{2/3})$. Bourgain gave a randomized construction for these which produces N frequencies in the interval $[1, e^{N^{1/3}}]$. By taking the inner product of the above sine sum with the conjugate Dirichlet kernel $D_M^* = \sum_{j=1}^{M} \sin jx$, whose L^1 norm is $c \log M$, we obtain that the number of frequencies in the interval $[1, M]$ is at most $cN^{2/3} \log M$, hence Bourgain's method does not go any further than it has to (in the size of the λ_j). One can even modify his randomized construction to produce $\lambda_j \sim e^{j^{1/3}}$. Hence the question becomes reasonable of whether it is just the growth of the frequencies that achieves the result. Our Theorem 1 answers this in the negative.

Bibliography

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